6.2 Radicals and Rational Exponents

Essential Question

How can you write and evaluate an *n*th root of a number?

Recall that you cube a number as follows.

3rd power

$2^3 = 2 \cdot 2 \cdot 2 = 8$ 2 cubed is 8.

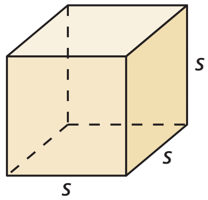
To "undo" cubing a number, take the cube root of the number.

Symbol for cube root is $\sqrt[3]{}$.

$\sqrt[3]{8} = \sqrt[3]{2^3} = 2$ The cube root of 8 is 2.

EXPLORATION 1 Finding Cube Roots

Work with a partner. Use a cube root symbol to write the side length of each cube. Then find the cube root. Check your answers by multiplying. Which cube is the largest? Which two cubes are the same size? Explain your reasoning.



a. Volume = 27 ft^3 **b.** Volume = 125 cm^3 **c.** Volume = 3375 $in.^3$

d. Volume = 3.375 m^3 **e.** Volume = 1 yd^3 **f.** Volume = $\frac{125}{8}$ mm^3

JUSTIFYING CONCLUSIONS

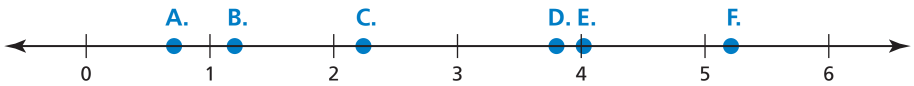
To be proficient in math, you need to justify your conclusions and communicate them to others.

EXPLORATION 2 Estimating *n*th Roots

Work with a partner. Estimate each positive *n*th root. Then match each *n*th root with the point on the number line. Justify your answers.

a. $\sqrt[4]{25}$ **b.** $\sqrt{0.5}$ **c.** $\sqrt[5]{2.5}$

d. $\sqrt[3]{65}$ **e.** $\sqrt[3]{55}$ **f.** $\sqrt[6]{20,000}$



Communicate Your Answer

3. How can you write and evaluate an *n*th root of a number?

4. The body mass *m* (in kilograms) of a dinosaur that walked on two feet can be modeled by

$$m = (0.00016)C^{2.73}$$

where *C* is the circumference (in millimeters) of the dinosaur's femur. The mass of a *Tyrannosaurus rex* was 4000 kilograms. Use a calculator to approximate the circumference of its femur.

6.2 Lesson

Core Vocabulary

*n*th root of *a*, *p. 300*
radical, *p. 300*
index of a radical, *p. 300*

Previous
square root

What You Will Learn

- Find *n*th roots.
- Evaluate expressions with rational exponents.
- Solve real-life problems involving rational exponents.

Finding *n*th Roots

You can extend the concept of a square root to other types of roots. For example, 2 is a cube root of 8 because $2^3 = 8$, and 3 is a fourth root of 81 because $3^4 = 81$. In general, for an integer n greater than 1, if $b^n = a$, then b is an ***n*th root of *a***. An nth root of a is written as $\sqrt[n]{a}$, where the expression $\sqrt[n]{a}$ is called a **radical** and n is the **index** of the radical.

You can also write an nth root of a as a power of a. If you assume the Power of a Power Property applies to rational exponents, then the following is true.

$$(a^{1/2})^2 = a^{(1/2) \cdot 2} = a^1 = a$$
$$(a^{1/3})^3 = a^{(1/3) \cdot 3} = a^1 = a$$
$$(a^{1/4})^4 = a^{(1/4) \cdot 4} = a^1 = a$$

Because $a^{1/2}$ is a number whose square is a, you can write $\sqrt{a} = a^{1/2}$. Similarly, $\sqrt[3]{a} = a^{1/3}$ and $\sqrt[4]{a} = a^{1/4}$. In general, $\sqrt[n]{a} = a^{1/n}$ for any integer n greater than 1.

READING

$\pm\sqrt[n]{a}$ represents both the positive and negative *n*th roots of *a*.

Core Concept

Real *n*th Roots of *a*

Let n be an integer greater than 1, and let a be a real number.

- If n is odd, then a has one real nth root: $\sqrt[n]{a} = a^{1/n}$
- If n is even and $a > 0$, then a has two real nth roots: $\pm\sqrt[n]{a} = \pm a^{1/n}$
- If n is even and $a = 0$, then a has one real nth root: $\sqrt[n]{0} = 0$
- If n is even and $a < 0$, then a has no real nth roots.

The nth roots of a number may be real numbers or *imaginary numbers*. You will study imaginary numbers in a future course.

EXAMPLE 1 Finding *n*th Roots

Find the indicated real nth root(s) of a.

a. $n = 3, a = -27$ **b.** $n = 4, a = 16$

SOLUTION

a. The index $n = 3$ is odd, so -27 has one real cube root. Because $(-3)^3 = -27$, the cube root of -27 is $\sqrt[3]{-27} = -3$, or $(-27)^{1/3} = -3$.

b. The index $n = 4$ is even, and $a > 0$. So, 16 has two real fourth roots. Because $2^4 = 16$ and $(-2)^4 = 16$, the fourth roots of 16 are $\pm\sqrt[4]{16} = \pm 2$, or $\pm 16^{1/4} = \pm 2$.

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Find the indicated real *n*th root(s) of *a*.

1. $n = 3, a = -125$ **2.** $n = 6, a = 64$

Evaluating Expressions with Rational Exponents

Recall that the radical $\sqrt{a}$ indicates the positive square root of a. Similarly, an nth root of a, $\sqrt[n]{a}$, with an *even* index indicates the positive nth root of a.

REMEMBER

The expression under the radical sign is the radicand.

EXAMPLE 2 **Evaluating *n*th Root Expressions**

Evaluate each expression.

a. $\sqrt[3]{-8}$ **b.** $-\sqrt[3]{8}$ **c.** $16^{1/4}$ **d.** $(-16)^{1/4}$

SOLUTION

a. $\sqrt[3]{-8} = \sqrt[3]{(-2) \cdot (-2) \cdot (-2)}$ Rewrite the expression showing factors.

$= -2$ Evaluate the cube root.

b. $-\sqrt[3]{8} = -\left(\sqrt[3]{2 \cdot 2 \cdot 2}\right)$ Rewrite the expression showing factors.

$= -(2)$ Evaluate the cube root.

$= -2$ Simplify.

c. $16^{1/4} = \sqrt[4]{16}$ Rewrite the expression in radical form.

$= \sqrt[4]{2 \cdot 2 \cdot 2 \cdot 2}$ Rewrite the expression showing factors.

$= 2$ Evaluate the fourth root.

d. $(-16)^{1/4}$ is not a real number because there is no real number that can be multiplied by itself four times to produce -16.

A rational exponent does not have to be of the form $1/n$. Other rational numbers such as $3/2$ can also be used as exponents. You can use the properties of exponents to evaluate or simplify expressions involving rational exponents.

STUDY TIP

You can rewrite $27^{2/3}$ as $27^{(1/3) \cdot 2}$ and then use the Power of a Power Property to show that

$27^{(1/3) \cdot 2} = (27^{1/3})^2$.

Core Concept

Rational Exponents

Let $a^{1/n}$ be an nth root of a, and let m be a positive integer.

Algebra $a^{m/n} = (a^{1/n})^m = \left(\sqrt[n]{a}\right)^m$

Numbers $27^{2/3} = (27^{1/3})^2 = \left(\sqrt[3]{27}\right)^2$

EXAMPLE 3 **Evaluating Expressions with Rational Exponents**

Evaluate (a) $16^{3/4}$ and (b) $27^{4/3}$.

SOLUTION

a. $16^{3/4} = (16^{1/4})^3$ Rational exponents

$= 2^3$ Evaluate the *n*th root.

$= 8$ Evaluate the power.

b. $27^{4/3} = (27^{1/3})^4$

$= 3^4$

$= 81$

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Evaluate the expression.

3. $\sqrt[3]{-125}$ **4.** $(-64)^{2/3}$ **5.** $9^{5/2}$ **6.** $256^{3/4}$

Solving Real-Life Problems

EXAMPLE 4 Solving a Real-Life Problem

Volume = 113 cubic feet

The radius r of a sphere is given by the equation $r = \left(\frac{3V}{4\pi}\right)^{1/3}$, where V is the volume of the sphere. Find the radius of the beach ball to the nearest foot. Use 3.14 for π.

SOLUTION

1. **Understand the Problem** You know the equation that represents the radius of a sphere in terms of its volume. You are asked to find the radius for a given volume.
2. **Make a Plan** Substitute the given volume into the equation. Then evaluate to find the radius.
3. **Solve the Problem**

$r = \left(\frac{3V}{4\pi}\right)^{1/3}$ Write the equation.

$= \left(\frac{3(113)}{4(3.14)}\right)^{1/3}$ Substitute 113 for V and 3.14 for π.

$= \left(\frac{339}{12.56}\right)^{1/3}$ Multiply.

≈ 3 Use a calculator.

▶ The radius of the beach ball is about 3 feet.

4. **Look Back** To check that your answer is reasonable, compare the size of the ball to the size of the woman pushing the ball. The ball appears to be slightly taller than the woman. The average height of a woman is between 5 and 6 feet. So, a radius of 3 feet, or height of 6 feet, seems reasonable for the beach ball.

EXAMPLE 5 Solving a Real-Life Problem

To calculate the annual inflation rate r (in decimal form) of an item that increases in value from P to F over a period of n years, you can use the equation $r = \left(\frac{F}{P}\right)^{1/n} - 1$. Find the annual inflation rate to the nearest tenth of a percent of a house that increases in value from \$200,000 to \$235,000 over a period of 5 years.

SOLUTION

$r = \left(\frac{F}{P}\right)^{1/n} - 1$ Write the equation.

$= \left(\frac{235{,}000}{200{,}000}\right)^{1/5} - 1$ Substitute 235,000 for F, 200,000 for P, and 5 for n.

$= 1.175^{1/5} - 1$ Divide.

≈ 0.03278 Use a calculator.

▶ The annual inflation rate is about 3.3%.

REMEMBER

To write a decimal as a percent, move the decimal point two places to the right. Then add a percent symbol.

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7. **WHAT IF?** In Example 4, the volume of the beach ball is 17,000 cubic inches. Find the radius to the nearest inch. Use 3.14 for π.
8. The average cost of college tuition increases from \$8500 to \$13,500 over a period of 8 years. Find the annual inflation rate to the nearest tenth of a percent.

6.2 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **WRITING** Explain how to evaluate $81^{1/4}$.

2. **WHICH ONE DOESN'T BELONG?** Which expression does *not* belong with the other three? Explain your reasoning.

$(\sqrt[3]{27})^2$	$27^{2/3}$	3^2	$(\sqrt[2]{27})^3$

Monitoring Progress and Modeling with Mathematics

In Exercises 3 and 4, rewrite the expression in rational exponent form.

3. $\sqrt{10}$

4. $\sqrt[5]{34}$

In Exercises 5 and 6, rewrite the expression in radical form.

5. $15^{1/3}$

6. $140^{1/8}$

In Exercises 7–10, find the indicated real *n*th root(s) of *a*. *(See Example 1.)*

7. $n = 2, a = 36$

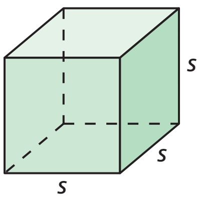
8. $n = 4, a = 81$

9. $n = 3, a = 1000$

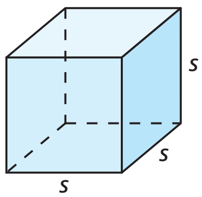
10. $n = 9, a = -512$

MATHEMATICAL CONNECTIONS **In Exercises 11 and 12, find the dimensions of the cube. Check your answer.**

11. Volume $= 64$ in.^{3}



12. Volume $= 216$ cm^3



In Exercises 13–18, evaluate the expression. *(See Example 2.)*

13. $\sqrt[4]{256}$

14. $\sqrt[3]{-216}$

15. $\sqrt[3]{-343}$

16. $-\sqrt[5]{1024}$

17. $128^{1/7}$

18. $(-64)^{1/2}$

In Exercises 19 and 20, rewrite the expression in rational exponent form.

19. $(\sqrt[5]{8})^4$

20. $(\sqrt[5]{-21})^6$

In Exercises 21 and 22, rewrite the expression in radical form.

21. $(-4)^{2/7}$

22. $9^{5/2}$

In Exercises 23–28, evaluate the expression. *(See Example 3.)*

23. $32^{3/5}$

24. $125^{2/3}$

25. $(-36)^{3/2}$

26. $(-243)^{2/5}$

27. $(-128)^{5/7}$

28. $343^{4/3}$

29. **ERROR ANALYSIS** Describe and correct the error in rewriting the expression in rational exponent form.

✗ $(\sqrt[3]{2})^4 = 2^{3/4}$

30. **ERROR ANALYSIS** Describe and correct the error in evaluating the expression.

✗
$$(-81)^{3/4} = [(-81)^{1/4}]^3$$
$$= (-3)^3$$
$$= -27$$

In Exercises 31–34, evaluate the expression.

31. $\left(\frac{1}{1000}\right)^{1/3}$

32. $\left(\frac{1}{64}\right)^{1/6}$

33. $(27)^{-2/3}$

34. $(9)^{-5/2}$

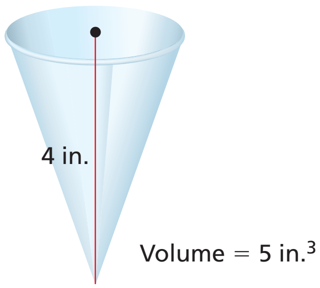
35. PROBLEM SOLVING A math club is having a bake sale. Find the area of the bake sale sign.

36. PROBLEM SOLVING The volume of a cube-shaped box is 27^5 cubic millimeters. Find the length of one side of the box.

37. MODELING WITH MATHEMATICS The radius r of the base of a cone is given by the equation

$$r = \left(\frac{3V}{\pi h}\right)^{1/2}$$

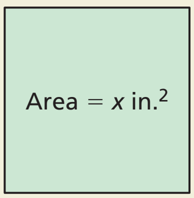
where V is the volume of the cone and h is the height of the cone. Find the radius of the paper cup to the nearest inch. Use 3.14 for π. *(See Example 4.)*



38. MODELING WITH MATHEMATICS The volume of a sphere is given by the equation $V = \frac{1}{6\sqrt{\pi}}S^{3/2}$, where S is the surface area of the sphere. Find the volume of a sphere, to the nearest cubic meter, that has a surface area of 60 square meters. Use 3.14 for π.

39. WRITING Explain how to write $(\sqrt[n]{a})^m$ in rational exponent form.

40. HOW DO YOU SEE IT? Write an expression in rational exponent form that represents the side length of the square.



In Exercises 41 and 42, use the formula $r = \left(\frac{F}{P}\right)^{1/n} - 1$ to find the annual inflation rate to the nearest tenth of a percent. *(See Example 5.)*

41. A farm increases in value from \$800,000 to \$1,100,000 over a period of 6 years.

42. The cost of a gallon of gas increases from \$1.46 to \$3.53 over a period of 10 years.

43. REASONING For what values of x is $x = x^{1/5}$?

44. MAKING AN ARGUMENT Your friend says that for a real number a and a positive integer n, the value of $\sqrt[n]{a}$ is always positive and the value of $-\sqrt[n]{a}$ is always negative. Is your friend correct? Explain.

In Exercises 45–48, simplify the expression.

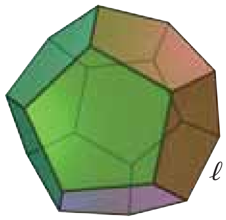
45. $(y^{1/6})^3 \cdot \sqrt{x}$

46. $(y \cdot y^{1/3})^{3/2}$

47. $x \cdot \sqrt[3]{y^6} + y^2 \cdot \sqrt[3]{x^3}$

48. $(x^{1/3} \cdot y^{1/2})^9 \cdot \sqrt{y}$

49. PROBLEM SOLVING The formula for the volume of a regular dodecahedron is $V \approx 7.66\ell^3$, where ℓ is the length of an edge. The volume of the dodecahedron is 20 cubic feet. Estimate the edge length.



50. THOUGHT PROVOKING Find a formula (for instance, from geometry or physics) that contains a radical. Rewrite the formula using rational exponents.

ABSTRACT REASONING In Exercises 51–56, let x be a nonnegative real number. Determine whether the statement is *always*, *sometimes*, or *never* true. Justify your answer.

51. $(x^{1/3})^3 = x$

52. $x^{1/3} = x^{-3}$

53. $x^{1/3} = \sqrt[3]{x}$

54. $x^{1/3} = x^3$

55. $\frac{x^{2/3}}{x^{1/3}} = \sqrt[3]{x}$

56. $x = x^{1/3} \cdot x^3$

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Evaluate the function when $x = -3, 0,$ and 8. *(Section 3.3)*

57. $f(x) = 2x - 10$

58. $w(x) = -5x - 1$

59. $h(x) = 13 - x$

60. $g(x) = 8x + 16$